So Called Vacuum Fluctuations as Correlation Functions
Mukhanov and Winitzki Chap 4 Notes

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Refs:

Introduction to Quantum Effects in Gravity, Mukhanov, V., and Winitzki, S. (Cambridge, 2007)

Student Friendly Quantum Field Theory, Klauber, R.D., (Sandtrove 2015, 2nd ed, 3rd printing)

1 The Correlation Function in Quantum Field Theory

1.1 What It Is Mathematically

In Mukhanov and Winitzki, pg. 51 unnumbered equation, and also, pg. 78, (6.48), the expectation value for vacuum fluctuations for a real scalar field ϕ is defined as a correlation function ε_ϕ, in the vacuum, between two different spatial locations $\mathbf{x}$ and $\mathbf{y}$ at the same time t.

$$\varepsilon_\phi(\mathbf{x}-\mathbf{y}) = \langle 0|\phi(\mathbf{x},t)\phi(\mathbf{y},t)|0\rangle . \tag{1}$$

Note that, if ϕ has local maxima at $\mathbf{x}$ and $\mathbf{y}$, ε_ϕ will have a higher value there than if it has a local maximum at $\mathbf{x}$ and close to a null value at $\mathbf{y}$. In the first case, the value of ϕ at $\mathbf{x}$ is highly correlated with that at $\mathbf{y}$, whereas in the second case, it is not. So, in general, $\varepsilon_\phi(\mathbf{x}-\mathbf{y})$ represents the correlation between different locations $\mathbf{x}$ and $\mathbf{y}$ of the field ϕ.

If ϕ represents a sinusoid, the highest correlations will occur between points separated by exact multiples of the wavelength of the sinusoid. A given peak is one wavelength from the next peak.

But generally ϕ is not a pure sinusoid. In any case, of course, if $\mathbf{x} = \mathbf{y}$, we would get maximum (full) correlation.

Note further, that (1) represents the vacuum expectation value (VEV) of the bilinear (correlation) operator $\phi(\mathbf{x},t)\phi(\mathbf{y},t)$. It is usually interpreted as the correlation one would expect to measure, on average, in the vacuum, between vacuum fields. We comment on this interpretation in Section 6.

Using ϕ as

$$\phi(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\int \frac{1}{\sqrt{2\omega_\mathbf{k}}}\left(a_\mathbf{k}e^{-ikx} + a^\dagger_\mathbf{k}e^{ikx}\right)d^3\mathbf{k} \qquad \text{Mukhanov and Winitzki (4.22), pg. 48} \tag{2}$$

in (1), we have

$$\varepsilon_\phi(\mathbf{x}-\mathbf{y}) = \langle 0|\left(\frac{1}{(2\pi)^{3/2}}\int \frac{1}{\sqrt{2\omega_\mathbf{k}}}\left(a_\mathbf{k}e^{-ikx} + a^\dagger_\mathbf{k}e^{ikx}\right)d^3\mathbf{k}\right)\left(\frac{1}{(2\pi)^{3/2}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}}\left(a_{\mathbf{k}'}e^{-ik'y} + a^\dagger_{\mathbf{k}'}e^{ik'y}\right)d^3\mathbf{k}'\right)|0\rangle . \tag{3}$$

1.2 What It Means in the Physical World

The correlation function (3) involves integration over all wave numbers $\mathbf{k}$ and hence may be visualized as correlating two points in physical space for a wave composed of all (an infinite number) wave length waves superimposed. Note that the amplitude of each wave is diminished by a factor of $1/\sqrt{\omega_\mathbf{k}}$. (For reasons why this is true of Klein-Gordon waves, see Klauber, pg. 47, "Normalization Factors" and "Relativistic Invariance of Probability" sections. Non-relativistic [Schroedinger or macro/classical] type waves do not have this factor.)

2 Deriving the Correlation Function

2.1 First Way to Derive Correlation Function

From (3), we have (where destruction operators destroy the vacuum, i.e., result in zero),

$$\begin{aligned}\varepsilon_\phi(\mathbf{x}-\mathbf{y}) &= \frac{1}{(2\pi)^3}\langle 0|\int \frac{1}{\sqrt{2\omega_\mathbf{k}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}}\begin{pmatrix} a_\mathbf{k}e^{-ikx}a_{\mathbf{k}'}e^{-ik'y} + a_\mathbf{k}e^{-ikx}a^\dagger_{\mathbf{k}'}e^{ik'y} \\ + a^\dagger_\mathbf{k}e^{ikx}a_{\mathbf{k}'}e^{-ik'y} + a^\dagger_\mathbf{k}e^{ikx}a^\dagger_{\mathbf{k}'}e^{ik'y}\end{pmatrix} d^3\mathbf{k}'d^3\mathbf{k}|0\rangle \\ &= \frac{1}{(2\pi)^3}\langle 0|\int \frac{1}{\sqrt{2\omega_\mathbf{k}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}}\begin{pmatrix} 0 + a_\mathbf{k}e^{-ikx}a^\dagger_{\mathbf{k}'}e^{ik'y} \\ + \quad 0 + a^\dagger_\mathbf{k}e^{ikx}a^\dagger_{\mathbf{k}'}e^{ik'y}\end{pmatrix} d^3\mathbf{k}'d^3\mathbf{k}|0\rangle .\end{aligned} \tag{4}$$

$$\begin{aligned}&= \frac{1}{(2\pi)^3}\langle 0|\int \frac{1}{\sqrt{2\omega_\mathbf{k}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}} a_\mathbf{k}e^{-ikx}a^\dagger_{\mathbf{k}'}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}|0\rangle \\ &\qquad + \frac{1}{(2\pi)^3}\underbrace{\langle 0|\int \frac{1}{\sqrt{2\omega_\mathbf{k}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}} e^{ikx}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}\left|\phi_k\phi_{k'}\right\rangle}_{=0 \text{ since bra and ket orthogonal}} .\end{aligned} \tag{5}$$

$$\begin{aligned}&= \frac{1}{(2\pi)^3}\langle 0|\int \frac{1}{\sqrt{2\omega_\mathbf{k}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}} a_\mathbf{k}a^\dagger_{\mathbf{k}'}e^{-ikx}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}|0\rangle \\ &= \frac{1}{(2\pi)^3}\langle 0|\int \frac{1}{\sqrt{2\omega_\mathbf{k}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}}\left(a^\dagger_{\mathbf{k}'}a_\mathbf{k} + \delta(\mathbf{k}-\mathbf{k}')\right)e^{-ikx}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}|0\rangle .\end{aligned} \tag{6}$$

Since $a_\mathbf{k}$ destroys the vacuum, we have

$$\begin{aligned}\varepsilon_\phi(\mathbf{x}-\mathbf{y}) &= \frac{1}{(2\pi)^3}\langle 0|\int \frac{1}{\sqrt{2\omega_\mathbf{k}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}}\delta(\mathbf{k}-\mathbf{k}')e^{-ikx}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}|0\rangle \\ &= \frac{1}{(2\pi)^3}\langle 0|\int \frac{1}{2\omega_\mathbf{k}}e^{-ik(x-y)}d^3\mathbf{k}|0\rangle = \frac{1}{(2\pi)^3}\int \frac{1}{2\omega_\mathbf{k}}\underbrace{e^{-i(\omega_\mathbf{k}-\omega_\mathbf{k})t}}_{=1}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}d^3\mathbf{k}\langle 0|0\rangle \\ &= \frac{1}{(2\pi)^3}\int \frac{1}{2\omega_\mathbf{k}}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}d^3\mathbf{k}.\end{aligned} \tag{7}$$

We evaluate (7) using polar coordinates in **k** space. (See Klauber, pg. 436, Fig. 17.4.) We now take the symbol $k = |\mathbf{k}|$, rather than the 4 momentum vector as it was used in exponents of (2) to (7), and θ is the angle between **k** and $(\mathbf{x}-\mathbf{y})$. Integration over k is from 0 to ∞, θ from 0 to 2π, and ϕ from 0 to π. Thus,

$$d^3\mathbf{k} = k^2 \sin\theta d\phi d\theta \qquad \mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) = k|\mathbf{x}-\mathbf{y}|\cos\theta . \tag{8}$$

With (8), (7) becomes

$$\varepsilon_\phi(\mathbf{x}-\mathbf{y}) = \frac{1}{(2\pi)^3}\int\frac{1}{2\omega_\mathbf{k}}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}d^3\mathbf{k} \xrightarrow[|\mathbf{x}-\mathbf{y}| \text{ dependence}]{(\mathbf{x}-\mathbf{y}) \text{ changes to}} \frac{1}{2(2\pi)^3}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{1}{\omega_\mathbf{k}}e^{ik|\mathbf{x}-\mathbf{y}|\cos\theta}k^2 d\phi \underbrace{\sin\theta d\theta}_{-d(\cos\theta)} dk$$

$$\begin{aligned}\varepsilon_\phi(|\mathbf{x}-\mathbf{y}|) &= \frac{-1}{2(2\pi)^3}2\pi\int_0^\infty\int_0^\pi\frac{k^2}{\omega_\mathbf{k}}e^{ik|\mathbf{x}-\mathbf{y}|\cos\theta}d\underbrace{(\cos\theta)}_{\text{take as } u}dk = \frac{-1}{2(2\pi)^2}\int_0^\infty\int_1^{-1}\frac{k^2}{\omega_\mathbf{k}}e^{ik|\mathbf{x}-\mathbf{y}|u}du\,dk \\ &= \frac{-1}{2(2\pi)^2}\int_0^\infty\frac{k^2}{\omega_\mathbf{k}}\left(\frac{e^{ik|\mathbf{x}-\mathbf{y}|u}}{ik|\mathbf{x}-\mathbf{y}|}\right)_1^{-1}dk = \frac{-1}{2(2\pi)^2}\int_0^\infty\frac{k^2}{\omega_\mathbf{k}}\left(\frac{e^{-ik|\mathbf{x}-\mathbf{y}|}-e^{ik|\mathbf{x}-\mathbf{y}|}}{ik|\mathbf{x}-\mathbf{y}|}\right)dk\end{aligned} \tag{9}$$

$$= \frac{1}{2(2\pi)^2}\int_0^\infty \frac{k^2}{\omega_{\mathbf{k}}}\left(\frac{e^{ik|\mathbf{x}-\mathbf{y}|}-e^{-ik|\mathbf{x}-\mathbf{y}|}}{ik|\mathbf{x}-\mathbf{y}|}\right)dk = \frac{1}{2(2\pi)^2}\left(\int_0^\infty \frac{k^2}{\omega_{\mathbf{k}}}\left(\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{ik|\mathbf{x}-\mathbf{y}|}\right)dk - \int_0^\infty \frac{k^2}{\omega_{\mathbf{k}}}\left(\frac{e^{-ik|\mathbf{x}-\mathbf{y}|}}{ik|\mathbf{x}-\mathbf{y}|}\right)dk\right)$$

$$= \frac{1}{2(2\pi)^2}\left(\int_0^\infty \frac{k^2}{\omega_{\mathbf{k}}}\left(\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{ik|\mathbf{x}-\mathbf{y}|}\right)(dk) - \int_0^{-\infty} \frac{k^2}{\omega_{\mathbf{k}}}\left(\frac{e^{-i(-k)|\mathbf{x}-\mathbf{y}|}}{i(-k)|\mathbf{x}-\mathbf{y}|}\right)(-dk)\right) \tag{10}$$

$$= \frac{1}{2(2\pi)^2}\left(\int_0^\infty \frac{k^2}{\omega_{\mathbf{k}}}\left(\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{ik|\mathbf{x}-\mathbf{y}|}\right)(dk) - \int_0^{-\infty} \frac{k^2}{\omega_{\mathbf{k}}}\left(\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{ik|\mathbf{x}-\mathbf{y}|}\right)dk\right)$$

$$= \frac{1}{2(2\pi)^2}\left(\int_0^\infty \frac{k^2}{\omega_{\mathbf{k}}}\left(\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{ik|\mathbf{x}-\mathbf{y}|}\right)(dk) + \int_{-\infty}^{0} \frac{k^2}{\omega_{\mathbf{k}}}\left(\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{ik|\mathbf{x}-\mathbf{y}|}\right)dk\right) \tag{11}$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^\infty \frac{k^2}{\omega_{\mathbf{k}}}\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{ik|\mathbf{x}-\mathbf{y}|}dk = \underbrace{\frac{1}{(2\pi)^2}\int_{-\infty}^\infty \frac{k^2}{\omega_{\mathbf{k}}}\left(\frac{\cos k|\mathbf{x}-\mathbf{y}|}{ik|\mathbf{x}-\mathbf{y}|}\right)dk}_{=0 \text{ since integrand is odd}} + \frac{1}{(2\pi)^2}\int_{-\infty}^\infty \frac{k^2}{\omega_{\mathbf{k}}}\left(\frac{i\sin k|\mathbf{x}-\mathbf{y}|}{ik|\mathbf{x}-\mathbf{y}|}\right)dk$$

or

$$\varepsilon_\phi(|\mathbf{x}-\mathbf{y}|) == \frac{1}{(2\pi)^2}\int_{-\infty}^\infty \frac{k^2}{\omega_{\mathbf{k}}}\left(\frac{\sin k|\mathbf{x}-\mathbf{y}|}{k|\mathbf{x}-\mathbf{y}|}\right)dk\ , \tag{12}$$

all of which, from (1) to (12), Mukhanov and Winitzki do in one step in the middle of page 51.

2.2 Second Way to Derive Correlation Function

From (5), we have

$$\varepsilon_\phi(\mathbf{x}-\mathbf{y}) = \frac{1}{(2\pi)^3}\langle 0|\int \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}}\ a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}e^{-ikx}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}|0\rangle\,. \tag{13}$$

<u>Aside</u>: Note that a ket in eigenstate $\mathbf{k}'$, in k space, is effectively a Dirac delta function with a peak at $\mathbf{k}'$.

$$|\phi_{\mathbf{k}'}\rangle = A\delta(\mathbf{k}-\mathbf{k}') \quad (\text{in } \mathbf{k} \text{ space basis}) \qquad A \text{ is a normalization factor}\,. \tag{14}$$

So, a number operator operating on (14) will leave the number of particles per unit $\mathbf{k}$ in the ket of 3-momentum, which is the delta function. That is,

$$\mathcal{N}_a(\mathbf{k})|\phi_{\mathbf{k}'}\rangle = a^\dagger_{\mathbf{k}}a_{\mathbf{k}}|\phi_{\mathbf{k}'}\rangle = \delta(\mathbf{k}-\mathbf{k}')|\phi_{\mathbf{k}'}\rangle\,. \tag{15}$$

Note that we have a discrete state/ket in (15) but we are working with the continuous fields case math. Via analogy with the discrete fields case[1], we can intuit that, given (15), then

Discrete Fields Case	Continuous Fields Case
$a^\dagger_{\mathbf{k}'}|0\rangle = \sqrt{1}|\phi_{\mathbf{k}'}\rangle$; $a^\dagger_{\mathbf{k}'}|n\phi_{\mathbf{k}'}\rangle = \sqrt{n+1}|(n+1)\phi_{\mathbf{k}'}\rangle$	$a^\dagger_{\mathbf{k}'}|0\rangle = \sqrt{\delta(\mathbf{k}-\mathbf{k}')}|\phi_{\mathbf{k}'}\rangle$. (16)
$a_{\mathbf{k}}|\phi_{\mathbf{k}}\rangle = \sqrt{1}|0\rangle$; $a_{\mathbf{k}}|n\phi_{\mathbf{k}}\rangle = \sqrt{n}|(n-1)\phi_{\mathbf{k}}\rangle$	$a_{\mathbf{k}}|\phi_{\mathbf{k}'}\rangle = \sqrt{\delta(\mathbf{k}-\mathbf{k}')}|0\rangle$

<u>End of aside</u>.

Thus, with (16) in (13) in we have

[1] See Klauber, pg. 59, (3-81).

$$\begin{aligned}\varepsilon_\phi(\mathbf{x}-\mathbf{y}) &= \frac{1}{(2\pi)^3}\langle 0|\int \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}}\; a_{\mathbf{k}} a^\dagger_{\mathbf{k}'}\, e^{-ikx}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}|0\rangle \\ &= \frac{1}{(2\pi)^3}\langle 0|\int \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}}\; a_{\mathbf{k}}\, e^{-ikx}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}\sqrt{\delta(\mathbf{k}-\mathbf{k}')}\,|\phi_{\mathbf{k}'}\rangle\end{aligned} \tag{17}$$

$$\begin{aligned}&= \frac{1}{(2\pi)^3}\langle 0|\int \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}}\, e^{-ikx}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}\sqrt{\delta(\mathbf{k}-\mathbf{k}')}\sqrt{\delta(\mathbf{k}-\mathbf{k}')}\,|0\rangle \\ &= \frac{1}{(2\pi)^3}\langle 0|\int \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}}\,\delta(\mathbf{k}-\mathbf{k}')e^{-ikx}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}|0\rangle,\end{aligned} \tag{18}$$

which is the same as we found in the first row of (7). The rest follows as before in (7) to (12).

2.3 Third Way to Derive Correlation Function

Starting again from (5), we have

$$\begin{aligned}\varepsilon_\phi(\mathbf{x}-\mathbf{y}) &= \frac{1}{(2\pi)^3}\langle 0|\int \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}}\; a_{\mathbf{k}} a^\dagger_{\mathbf{k}'}\, e^{-ikx}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}|0\rangle \\ &= \frac{1}{(2\pi)^3}\langle 0|\int \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}}\; a_{\mathbf{k}}\, e^{-ikx}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}\sqrt{\delta(\mathbf{k}-\mathbf{k}')}\,|\phi_{\mathbf{k}'}\rangle,\end{aligned} \tag{19}$$

but now consider operating on the bra (not the ket) with $a_{\mathbf{k}}$. This yields

$$\begin{aligned}\varepsilon_\phi(\mathbf{x}-\mathbf{y}) &= \frac{1}{(2\pi)^3}\langle \phi_{\mathbf{k}}|\sqrt{\delta(\mathbf{k}'-\mathbf{k})}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}} e^{-ikx}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}\sqrt{\delta(\mathbf{k}-\mathbf{k}')}\,|\phi_{\mathbf{k}'}\rangle \\ &= \frac{1}{(2\pi)^3}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}}\sqrt{\delta(\mathbf{k}-\mathbf{k}')}\sqrt{\delta(\mathbf{k}-\mathbf{k}')}\, e^{-ikx}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}\underbrace{\langle \phi_{\mathbf{k}}|\phi_{\mathbf{k}'}\rangle}_{\delta_{\mathbf{k}\mathbf{k}'}} \\ &= \frac{1}{(2\pi)^3}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\int \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}}\,\delta(\mathbf{k}-\mathbf{k}')e^{-ikx}e^{ik'y}d^3\mathbf{k}'d^3\mathbf{k}.\end{aligned} \tag{20}$$

which is the same as we found in the first row of (7). The rest follows as before in (7) to (12).

2.4 Fourth Way to Derive the Correlation Function

Note that (1) is simply the particle form of the propagator as shown in Klauber, pg. 72, (3-116) and elsewhere, where $t_y = t_x = t$, i.e.,

$$\begin{aligned}&\text{for } t_y \le t_x \;\; \text{(particle)} \quad \langle 0|\phi(x)\phi^\dagger(y)|0\rangle \qquad (3-116) \text{ and } (3-118) \text{ in Klauber} \\ &\qquad \text{where in the present case } \phi \text{ is real, i.e., } \phi^\dagger = \phi.\end{aligned} \tag{21}$$

In the reference, this is shown to equal

$$i\Delta^+(x-y) = \underbrace{\langle 0|\phi(x)\phi^\dagger(y)|0\rangle}_{\text{for } t_y \le t_x} = \frac{1}{2(2\pi)^3}\int \frac{e^{-ik(x-y)}}{\omega_{\mathbf{k}}}d^3\mathbf{k} \qquad (3-129) \text{ in Klauber} \tag{22}$$

which for, equal times, as in (1) where $t_y = t_x = t$, is simply

$$= \frac{1}{2(2\pi)^3}\int \frac{\overbrace{e^{-i(\omega_{\mathbf{k}}t-\omega_{\mathbf{k}}t)}}^{=e^0=1}\, e^{+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\omega_{\mathbf{k}}}d^3\mathbf{k} = \varepsilon_\phi(\mathbf{x}-\mathbf{y}), \tag{23}$$

as expressed in the last row of (7).

3 Evaluating the Correlation Function

We can consider the value of (12), repeated below, for convenience.

$$\varepsilon_\phi\left(|\mathbf{x}-\mathbf{y}|\right) == \frac{1}{(2\pi)^2}\int \frac{k^2}{\omega_{\mathbf{k}}}\left(\frac{\sin k|\mathbf{x}-\mathbf{y}|}{k|\mathbf{x}-\mathbf{y}|}\right)dk\,, \tag{12}$$

3.1 Massless Field Case

Since

$$\omega_{\mathbf{k}}^2 - k^2 = m^2 \qquad \text{where } k = \pm|\mathbf{k}| = \pm\omega_{\mathbf{k}}\,, \tag{24}$$

for the massless case, (12) becomes

$$\begin{aligned}
\varepsilon_\phi\left(|\mathbf{x}-\mathbf{y}|\right) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\frac{k}{|\mathbf{k}|}\frac{\sin k|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}-\mathbf{y}|}dk \qquad \text{for } m=0\text{, i.e., } |\mathbf{k}| = \omega_{\mathbf{k}} \\
&= \frac{2}{(2\pi)^2}\int_0^{+\infty}\frac{\sin k|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}-\mathbf{y}|}dk = \frac{1}{2\pi^2|\mathbf{x}-\mathbf{y}|}\int_0^{+\infty}\sin k|\mathbf{x}-\mathbf{y}|dk = \frac{-1}{2\pi^2|\mathbf{x}-\mathbf{y}|}\frac{\cos k|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}-\mathbf{y}|}\Bigg|_{k=0}^{k=\infty} \\
&= \frac{1}{2\pi^2|\mathbf{x}-\mathbf{y}|^2}\left(1-\cos\infty\right).
\end{aligned} \tag{25}$$

The cosine of infinity term can vary between –1 and +1 and depends on where (what k value) at very large k values we choose to cutoff the cosine oscillation and call it the end. (Its average value for a large number of random endpoints is zero.) If we just ignore that term, we get an effective evaluation of the correlation between two points **x** and **y**.

$$\varepsilon_\phi\left(|\mathbf{x}-\mathbf{y}|\right) = \frac{1}{2\pi^2|\mathbf{x}-\mathbf{y}|^2} = \frac{1}{2\pi^2 L^2} \qquad m=0 \qquad L = |\mathbf{x}-\mathbf{y}| \tag{26}$$

The correlation varies quadratically with the inverse of L, the distance between **x** and **y**. Note when **x = y**, the correlation is infinite, which makes some sense since a field at any point is completely correlated with itself at that point (at the same time).

3.2 Estimate for the Massive Field Case

Relation (12) is not so easy to evaluate when $m \neq 0$, as then we have

$$\begin{aligned}
\varepsilon_\phi\left(|\mathbf{x}-\mathbf{y}|\right) &= \frac{1}{(2\pi)^2}\int\frac{k^2}{\sqrt{k^2+m^2}}\left(\frac{\sin k|\mathbf{x}-\mathbf{y}|}{k|\mathbf{x}-\mathbf{y}|}\right)dk \\
&= \frac{1}{(2\pi)^2|\mathbf{x}-\mathbf{y}|}\int\frac{k}{\sqrt{k^2+m^2}}\sin k|\mathbf{x}-\mathbf{y}|dk = \frac{1}{(2\pi)^2 L}\int\frac{k}{\sqrt{k^2+m^2}}\sin kL\,dk \qquad L = |\mathbf{x}-\mathbf{y}|.
\end{aligned}\,, \tag{27}$$

and there doesn't seem to be an integration table formula available for (27). Numerical integration appears in order for a precise solution at given L. However, we can get an estimate for (27), and Fig. 1 can help demonstrate how.

In the top row of Fig. 1, where m = 0, we can see that in the integration, all but the shaded area on the right side of the vertical axis will cancel out (ignoring the arbitrary cutoff we might chose on the far right and far left to approximate infinity). For the lower rows of Fig. 1, the shaded region just to the right of the vertical axis approximates the total integral value, i.e., it is of the same order. So our job is to approximate the total integral by getting a good estimate of the region of integration of the curve from 0 to π.

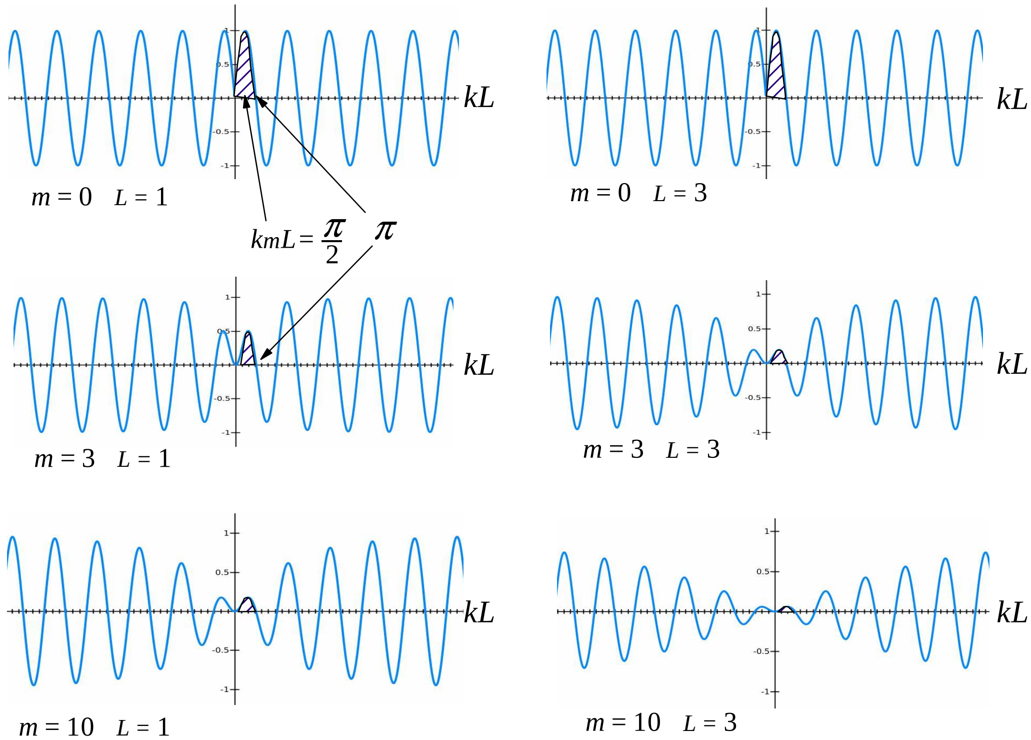

Figure 1. Plots of the Integrand $\dfrac{k}{\sqrt{k^2+m^2}}\,sinkL$ **vs** kL

In the top row of the figure, the curve from 0 to π is half of a pure sine wave. In the lower rows, a half sine wave is a reasonable (of the same order) approximation over that range. From the figure, we can see the amplitude of that approximate since wave is reduced for higher m or L. Note that the integration in (27) is over k, but to simplify, the plots of Fig. 1 are shown as over kL. So given our approximations, we will estimate the integral of the entire integrand in (27) as that of a sine wave over k from kL of 0 to π, whose amplitude varies with m or L.

To start, we will use the symbol k_f to denote the value of k when $kL = \pi$, and note that the integral over k in the region of interest can be estimated with the help of the following.

$$k_f L = \pi \qquad \rightarrow \qquad \int_0^{k_f = \frac{\pi}{L}} \sin kL\,dk = \left.\frac{-\cos kL}{L}\right|_0^{\frac{\pi}{L}} = \frac{1}{L}\left(-\cos\pi + \cos 0\right) = \frac{2}{L}. \tag{28}$$

For a sine curve of amplitude 1, as in the top row of Fig. 1, (28) gives us the area under the curve from $kL = 0$ to π (k from zero to π/L). For other amplitudes, as in the 2nd and 3rd rows of Fig. 1, we need to multiply (28) by the maximum value (the sine curve amplitude) of the integrand, which is the value of the amplitude at the mid point where $kL = \pi/2$. We label the k value at this point as k_m. (The subscript m can stand for (local) maximum, mid-point, or main contributing wave number to the area.) So,

$$k_m L = \frac{\pi}{2} \quad \rightarrow \quad k_m = \frac{\pi}{2L}\ . \tag{29}$$

Thus, the height of the function (integrand of (27) at k_m is the

$$\text{estimate of sine curve amplitude} = \frac{k_m}{\sqrt{\left(k_m\right)^2 + m^2}}\,. \tag{30}$$

So using (28), (29), and (30) to estimate (27), we have

$$\varepsilon_\phi(L) = \frac{1}{(2\pi)^2 L}\int_{-\infty}^{+\infty} \frac{k}{\sqrt{k^2+m^2}} \sin kL\, dk \approx \frac{1}{(2\pi)^2 L}\int_0^{\frac{\pi}{L}} \frac{k_m}{\sqrt{(k_m)^2+m^2}} \sin kL\, dk$$
$$= \frac{1}{(2\pi)^2 L}\frac{k_m}{\sqrt{(k_m)^2+m^2}}\int_0^{\frac{\pi}{L}} \sin kL\, dk = \frac{1}{(2\pi)^2 L}\frac{\frac{\pi}{2L}}{\sqrt{\left(\frac{\pi}{2L}\right)^2+m^2}}\frac{2}{L}, \tag{31}$$

or finally,

$$\varepsilon_\phi(L) \approx \frac{1}{2\pi^2 L^2}\frac{1}{\sqrt{1+\left(\frac{2mL}{\pi}\right)^2}}. \tag{32}$$

Note this checks with our earlier result (26) for $m = 0$, as it should since our assumption of a pure sine wave over the region $kL = 0$ to π is precisely correct in that case.

3.3 Conclusions for Behavior

3.3.1 With respect to *m, L*

Note for $m << L$ (large separation and/or small mass), ε_ϕ effectively varies inversely with the square of L. For $m >> L$, it effectively varies with the inverse of the cube of L. As $L \to 0$, $\varepsilon_\phi(L) \to \infty$, and the correlation function, as defined in (1), is infinite.

3.3.2 With respect to wave number

Note from Fig. 1 that of all waves, the largest (main) contributor to the correlation is the one with wave number k_m. And, we can relate ε_ϕ to k_m, instead of L, via use of (29) in (32), i.e.,

$$\varepsilon_\phi(L) \approx \frac{1}{2\pi^2 L^2}\frac{1}{\sqrt{1+\left(\frac{2mL}{\pi}\right)^2}} \xrightarrow{L=\frac{\pi}{2k_m}} \varepsilon_\phi(L) \approx \frac{1}{2\pi^2\left(\frac{\pi}{2k_m}\right)^2}\frac{1}{\sqrt{1+\left(\frac{2m}{\pi}\frac{\pi}{2k_m}\right)^2}}, \tag{33}$$

or

$$\varepsilon_\phi(L) \approx \frac{2k_m^2}{\pi^4\sqrt{1+\left(\frac{m}{k_m}\right)^2}}. \tag{34}$$

3.3.3 With respect to *m* and k_m

One commonly sees, as in Mukhanov and Winitzki, pg. 51, (4.34), k_m taken as simply k in (34). Given that, one can surmise that for $m << k$ (k_m really), ε_ϕ effectively varies directly with the square of k_m; for $m >> k$ (k_m really), directly with the cube of k_m.

All of this Section 3, from pg. 5 to pg. 7, is done in one step in Mukhanov and Winitzki in eq. (4.34).

4 Correlation Function for Discrete States

Consider the discrete real field (which has finite volume boundary conditions)

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(a(\mathbf{k})e^{-ikx} + a^{\dagger}(\mathbf{k})e^{ikx} \right) \tag{35}$$

in the correlation function (1) (repeated below for convenience)

$$\varepsilon_{\phi}\left(\mathbf{x}-\mathbf{y}\right) = \left\langle 0 \middle| \phi\left(\mathbf{x},t\right)\phi\left(\mathbf{y},t\right) \middle| 0 \right\rangle . \tag{1}$$

The effect of the creation/destruction operators is the same as for the continuous case of (3) through (6) except the commutation relations give us a Kronecker delta instead of the Dirac delta function.

$$\varepsilon_{\phi}\left(\left|\mathbf{x}-\mathbf{y}\right|\right) = \frac{1}{V}\left\langle 0\right| \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \sum_{\mathbf{k}'} \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}} \left(a^{\dagger}_{\mathbf{k}'} a_{\mathbf{k}} + \delta_{\mathbf{k}\mathbf{k}'} \right) e^{-ikx} e^{ik'y} \left| 0 \right\rangle . \tag{36}$$

As with the parallel continuous case, since $a_{\mathbf{k}}$ destroys the vacuum, we have

$$\begin{aligned} \varepsilon_{\phi}\left(\mathbf{x}-\mathbf{y}\right) &= \frac{1}{V}\left\langle 0\right| \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \sum_{\mathbf{k}'} \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}} \delta_{\mathbf{k}\mathbf{k}'} e^{-ikx} e^{ik'y} \left| 0 \right\rangle \\ &= \frac{1}{V}\left\langle 0\right| \sum_{\mathbf{k}} \frac{1}{2\omega_{\mathbf{k}}} e^{-ikx} e^{ik'y} \left| 0 \right\rangle = \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{2\omega_{\mathbf{k}}} \underbrace{e^{-i\left(\omega_{\mathbf{k}}-\omega_{\mathbf{k}}\right)t}}_{=1} e^{i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{y}\right)} \left\langle 0 \middle| 0 \right\rangle \\ &= \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{2\omega_{\mathbf{k}}} e^{i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{y}\right)}, \end{aligned} \tag{37}$$

which parallels (7). In fact, as is well known, in the limit $V \to \infty$ in the last row of (37), with appropriate boundary conditions, we have

$$\varepsilon_{\phi}\left(\mathbf{x}-\mathbf{y}\right) = \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{2\omega_{\mathbf{k}}} e^{i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{y}\right)} \xrightarrow[\text{discrete to continuous}]{V\to\infty} \varepsilon_{\phi}\left(\mathbf{x}-\mathbf{y}\right) = \frac{1}{\left(2\pi\right)^{3}} \int \frac{1}{2\omega_{\mathbf{k}}} e^{i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{y}\right)} d^{3}\mathbf{k} , \tag{38}$$

which corroborates our previous result (7).

5 More on the Physical Interpretation of the Correlation Function

Fig. 2 can help us to get a better feeling for what we are really measuring with our correlation function $\varepsilon_{\phi}(L)$.

Note from the LHS of Fig. 2, we could find a correlation function for a single plane wave, if we desired. But that is not what we have done in prior sections, which is represented by the RHS of the figure.

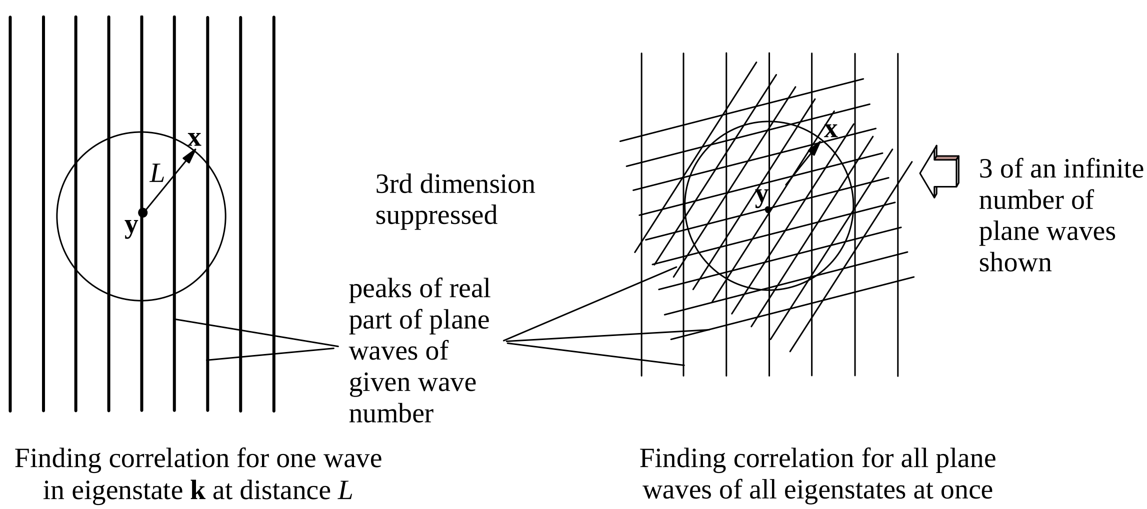

Figure 2. Showing What We Are Really Finding with $\varepsilon_{\phi}(L)$ (The RHS Above)

We have started with the ϕ of (2), which is a superposition of an infinite number of plane waves, in all directions, with all possible wave numbers $|\mathbf{k}|$. That is what we found in (32) and other expressions for $\varepsilon_\phi(L)$. We are determining the correlation between any two points $\mathbf{x}$ and $\mathbf{y}$, given the form of ϕ in (2), an integration over all possible waves of eigenvector $\mathbf{k}$. Each eigenwave is modulated only by the factor $1/\sqrt{\omega_\mathbf{k}}$ and otherwise all such waves contribute equally. Note that $\varepsilon_\phi(L)$ is the same for any two points a distance L apart.

6 Interpretation of Correlation Function for Vacuum

Note several things.

1. The correlation function $\varepsilon_\phi\left(|\mathbf{x}-\mathbf{y}|\right)$ is independent of time. i.e., it is *not* fluctuating. It is the same for any time t. It may be hard to see how it can represent "fluctuations" occurring in time in the vacuum or anywhere, as vacuum fluctuations are often portrayed. But the underlying field ϕ from which we determine ε_ϕ is oscillating (fluctuating.)

It may be argued to behave like a typical plane wave function for a free particle in non-relativistic quantum mechanics (NRQM) of form

$$\psi_{state} = A_\mathbf{k} e^{-i\omega_\mathbf{k} t} e^{+i\mathbf{k}\cdot\mathbf{x}}, \tag{39}$$

which varies in time, but the probability density

$$\psi^\dagger_{state}\psi_{state} = A^\dagger_\mathbf{k} A_\mathbf{k} = \text{ constant in time}. \tag{40}$$

That is, the particle/state itself oscillates in time, but anything we might detect like particle position probability does not. In fact, phase (such as would change with time) is irrelevant with respect to any measurements, so there seems to be little chance of detecting any sort of fluctuation in time of states existing in the vacuum. If they cannot be detected, one could argue that, for all practical purposes, they are not there.

2. The correlation function represents the correlation between locations in a particular wave/state/particle ϕ (in the vacuum or otherwise). It is not related to a pair of particles, which might be popping into or out of the vacuum, as is commonly depicted. Not only is the correlation function representative of a single particle (comprised of all possible eigenstates superimposed), rather than a pair, it is static. It does not pop in and out of existence over time.

3. What seems to be happening in determining the correlation function appears different in each of the different approaches of Sections 2.1 to 2.4. The interpretation in each case is different.

<u>First Way to Derive Correlation Function</u>

In Section 2.1, our ket and bra remain vacuum states, and a non-zero correlation function arises in (6) from the operator commutation relation. That gives us a delta function, which is a number and has no effect on the vacuum ket or bra. The delta function gives us the result of (7).

So one could argue we are determining the correlation function for the vacuum, and if this is non-zero, it implies there are fields/particles in the vacuum. A pure zero number of fields in the vacuum should give rise to a zero correlation function.

Indeed, the ½ quanta of the vacuum arise from the non-commutation relations as well. (See Klauber, pg. 54, (3-54).) So the source of the infinite (or at least enormous) energy in the vacuum and the source of the correlation function are the same, the non-commutation of creation/destruction operators.[2]

[2] The means for elimination of the ½ quanta energy in the vaccum suggested by Klauber (footnote pg. 50) also eliminates vacuum fluctuations. That is, it yields a zero value for (1). It simply incorporates mathematically valid solutions to the field equations (Klein-Gordon here) that have not been previously incorporated into QFT.

Second Way to Derive Correlation Function

In Section 2.2, we do not use the commutation relations to convert the $a_{\mathbf{k}}a^{\dagger}_{\mathbf{k}'}$ to $a^{\dagger}_{\mathbf{k}'}a_{\mathbf{k}}$ plus a delta function. Instead we simply act directly on the vacuum with $a_{\mathbf{k}}a^{\dagger}_{\mathbf{k}'}$. See (17) and (18). This implies we are creating a particle then destroying it all at the same time *t*. We get the same result.

From this perspective, we are not measuring the correlation of fields in the vacuum, but the correlation between the state we create and the state we destroy, which are the same. That is, the correlation function represents a correlation of a non-vacuum field ϕ with itself, and is not a measure of what is going on in the vacuum.

Third Way to Derive Correlation Function

In Section 2.3, we again do not use the commutation relations, but employ $a_{\mathbf{k}}a^{\dagger}_{\mathbf{k}'}$ to create a ket state via $a^{\dagger}_{\mathbf{k}'}$ and a create a bra state via $a_{\mathbf{k}}$. See (19) and (20).

From this perspective, we are again not measuring the correlation of fields in the vacuum, but the correlation between a created state and itself. That is, it is not a measure of what is going on in the vacuum.

Fourth Way to Derive Correlation Function

In Section 2.4, the same correlation function is seen to be simply the equal times Feynman propagator for a real field (both events are at time *t*). See (21) to (23).

Propagators arise naturally in QFT via the Dyson-Wicks expansion, and represent virtual particles mediating interactions between real particles[3]. Such propagators do not simply pop in and out of the vacuum all alone, but are always linked to other real particles in interactions[4].

Thus, one could argue that the correlation function, as defined in (1), does not represent a characteristic of the vacuum existing in the absence of real world particles.

As an aside, the correlation function (1) with non equal times is not static but fluctuates in time. It is, of course, simply the Feynman propagator. This may be the reason propagators are sometimes referred to as vacuum fluctuations, or correlation functions.

Conclusion: There are other interpretations to $\langle 0|\phi\phi|0\rangle$ than that of a correlation function arising from vacuum fluctuations.

4. Consider, instead of the real field of (2), which is chargeless (i.e., the particle is its own antiparticle), a complex field (the particle has a distinct antiparticle),which has charge. Should the correlation function be $\phi^{\dagger}\phi$ or $\phi\phi$?

One would expect the latter, as it is correlating the field with itself, rather than its complex conjugate. But that ends up with $(a_{\mathbf{k}} + b_{\mathbf{k}}^{\dagger})(a_{\mathbf{k}} + b_{\mathbf{k}}^{\dagger})$ type terms in the correlation function (1), rather than $(a_{\mathbf{k}} + a_{\mathbf{k}}^{\dagger})(a_{\mathbf{k}}^{\dagger} + a_{\mathbf{k}})$ type. That is, from (1),

$$\langle 0|\phi\phi|0\rangle \quad \rightarrow \quad \text{terms of form } \langle 0|\left(a^{\dagger}_{\mathbf{k}}+b_{\mathbf{k}}\right)\left(a^{\dagger}_{\mathbf{k}}+b_{\mathbf{k}}\right)|0\rangle = \langle 0|\left(a^{\dagger}_{\mathbf{k}}a^{\dagger}_{\mathbf{k}}+a^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}+b_{\mathbf{k}}a^{\dagger}_{\mathbf{k}}+b_{\mathbf{k}}b_{\mathbf{k}}\right)|0\rangle = 0 \qquad (41)$$

In that case all terms have $a_{\mathbf{k}}a_{\mathbf{k}}$, $a_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}$, $b_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}$ or $b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}^{\dagger}$ that either destroy the vacuum (leave zero) or form a ket different from the bra, thereby ending up with zero. That is, there is zero vacuum correlation for a complex field (associated with particles like electrons and quarks).

[3] The term *real field* means the field is a real, not complex, function of space and time. The term *real particle* means a particle that is not virtual.
[4] We are ignoring "vacuum bubbles" that do arise naturally from the Dyson-Wicks expansion. See Klauber, Section 8.4.8, pg. 234. Such vacuum bubbles always consist of three virtual particles, however, not a single one, as ε_{ϕ} relates to.

Additionally, nothing like $\phi\phi$ (for complex fields) appears anywhere in QFT. Not in the free theory; and not in the interacting theory. It plays no role in determining the transition amplitude for real world interactions. The transition amplitude, derived from the state equation of motion, has no $\phi\phi$ terms in it.

Conclusion: There is no correlation for a complex (charged) field with itself, as determined by (1), i.e., $\langle 0|\phi\phi|0\rangle$. Further, $\phi\phi$ plays no role in any transition amplitude.

On the other hand, if we were to use $\phi^\dagger\phi$ instead of $\phi\phi$ in (1), then we are not correlating a field with itself, but with its complex conjugate. Is that a meaningful correlation?

Still further, if we were to evaluate $\langle 0|\phi^\dagger\phi|0\rangle$, we would find a correlation only between *b* type particles and not *a* type. That is,

$$\langle 0|\phi^\dagger\phi|0\rangle \quad \rightarrow \quad \text{terms of form } \langle 0|\left(a_{\mathbf{k}}^\dagger + b_{\mathbf{k}}\right)\left(a_{\mathbf{k}} + b_{\mathbf{k}}^\dagger\right)|0\rangle = \langle 0|\left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger + b_{\mathbf{k}} a_{\mathbf{k}} + b_{\mathbf{k}} b_{\mathbf{k}}^\dagger\right)|0\rangle . \tag{42}$$

All of these yield zero except the $b_{\mathbf{k}} b_{\mathbf{k}}^\dagger$ term. So, no matter which of the ways of Section 2 we use to evaluate $\langle 0|\phi^\dagger\phi|0\rangle$, the result will only apply to *b* type particles and not *a* type, i.e., only to antiparticles and not particles. (Similarly, $\langle 0|\phi\phi^\dagger|0\rangle$ would only apply to particles and not antiparticles.)

Conclusion: It is hard to conclude we get a meaningful correlation function from $\langle 0|\phi^\dagger\phi|0\rangle$ if only antiparticles, and not particles, are correlated.

5. Wouldn't we want the correlation to be with respect to probability density at various points **x** and **y**? Isn't the important thing how correlated what we measure (detect the particle) is, rather than something we can't measure (the field itself)?

If we are talking about what we measure, should we be correlating probability density ρ(**x**, t) with ρ(**y**, t), rather than ϕ(**x**, t) with ϕ(**y**, t)? For a relativistic quantum scalar field theory, this is (43). (See Klauber, Sect. 3.7 to Sect. 3.81., pgs 61- 64, and Sect. 5.6, (5-62), pg. 149.)

$$\rho = i\left(\dot{\phi}\phi^\dagger - \dot{\phi}^\dagger\phi\right) \tag{43}$$

For a real field, ρ is zero everywhere and for all time, and thus it is surely so for the vacuum. This led to the interpretation of (43) as charge density.

For a complex field, we get, for a single particle state (Klauber, pg. 62, (3-91))

$$\rho = \frac{1}{V}\sum_{\mathbf{k}} \left(N_a(\mathbf{k}) - N_b(\mathbf{k})\right) \qquad \rightarrow \qquad \langle \phi_{\mathbf{k}}|\rho|\phi_{\mathbf{k}}\rangle = \frac{1}{V}\ , \tag{44}$$

where there are no ½ terms in the probability (charge) density ρ, so there is no contribution from the vacuum. Thus,

$$\langle 0|\rho|0\rangle = 0 \qquad \rightarrow \qquad \langle 0|\rho(\mathbf{x},t)\rho(\mathbf{y},t)|0\rangle = 0 \quad . \tag{45}$$

Conclusion: For the vacuum, there is no correlation between probability densities, i.e., from one point to another in what we should be able to measure experimentally.

6. $\phi(\mathbf{x},t)\phi(\mathbf{y},t)$ is not a measurable, in the usual quantum sense. $\phi(\mathbf{x},t)\phi(\mathbf{y},t)$ implies we can measure some quantity that is evaluated at two different points. But all measurables in QFT, like energy density, 3-momentum density, charge density, etc are measurable at a single point, i.e., related to $\phi(\mathbf{x},t)\phi(\mathbf{x},t)$ at the same **x** at the same time *t*. Both field factors in any QFT bilinear operator similar in form to $\phi\phi$ are evaluated at same event for measurable quantities.

Conclusion: $\phi(\mathbf{x},t)\phi(\mathbf{y},t)$ is not a quantum measurable/observable.

7. It is interesting that, for the massless case of (26), the correlation falls off like the photon radiation intensity from a charged point source radiating in all directions isotropically. I am not sure what this might imply.

8. As an aside, one would think the vacuum could not have any states in it, or $a_{\mathbf{k}}|0\rangle$would not = 0.

7 Possible Conclusions

From Section 6, reasonable arguments could be made that the correlation function for a real scalar field ϕ, $\varepsilon_\phi(\mathbf{x}-\mathbf{y}) = \langle 0|\phi(\mathbf{x},t)\phi(\mathbf{y},t)|0\rangle$, does not represent fluctuations in fields residing in the vacuum, since there are different ways to derive the final result, which comprise creation of states.

For complex fields ϕ, i.e., fields associated with charged particles, the different possible definitions of ε_ϕ lead to 1) zero (for $\varepsilon_\phi = \langle 0|\phi(\mathbf{x},t)\phi(\mathbf{y},t)|0\rangle$), 2) inclusion of antiparticles, but not particles (for $\varepsilon_\phi(\mathbf{x}-\mathbf{y}) = \langle 0|\phi^\dagger(\mathbf{x},t)\phi(\mathbf{y},t)|0\rangle$), or 3) inclusion of particles, but not antiparticles (for $\varepsilon_\phi(\mathbf{x}-\mathbf{y}) = \langle 0|\phi(\mathbf{x},t)\phi^\dagger(\mathbf{y},t)|0\rangle$).

Correlation of probability densities at two locations **x** and **y**, at the same time *t* in the vacuum, yields zero. This is true for any form of ϕ, not just that of (2). That is, it would be true if, for example, ϕ represented a Gaussian wave packet. There is no vacuum correlation for probability density. In fact, probability density for the vacuum is identically zero.

Further arguments could also be made that ε_ϕ does not represent a measurable quantity.